COMPUTER SEARCH FOR CURVES WITH MANY POINTS AMONG UNRAMIFIED COVERS OF GENUS 2 CURVES OVER $\mathbf{F}_5$, $\mathbf{F}_7$, $\mathbf{F}_9$ AND $\mathbf{F}_{11}$

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Abstract.[1] Using class field theory one associates to each curve C over a finite field, and each subgroup G of its class group, an unramified cover of C whose genus is determined by the index of G. By listing class groups of curves of small genus one may get examples of curves with many points. For example, in [1] this search was done for small fields of characteristic 2 and 3, and for base curves of genus between 2 and 5; we do it for all base curves of genus 2 over the fields of cardinality 5 and 7, and for some base curves of genus 2 over the fields of cardinality 9 and 11, giving new entries for the tables of curves with many points [4].

Background

For q a prime power and g a non-negative integer, let $N_q(g)$ the maximum possible number of rational places on a global function field of genus g over $\mathbf{F}_q$. Much work has been done to determine $N_q(g)$ for different values of q and g; the current intervals in which $N_q(g)$ in known to lie together with relevant references are given in the tables [4]. In this report we improve upon the lower bounds by constructing curves with many points for some genera over the fields with $q = 5, 7, 9$ and 11 elements. Let us note that in order for a curve to be considered to have many points it has to have more than $b(q, g)/\sqrt{2}$ points, where $b(q, g)$ is a known upper bound of $N_q(g)$, see [3].

The method

Attaching an unramified cover to a subgroup of the class group: Let C be a smooth, geometrically irreducible, projective curve of genus g defined over the finite field k and let $\mathrm{Jac}(C)$ be its Jacobian variety. If G is a subgroup of $J := \mathrm{Jac}(C)(k)$ of index d then class field theory gives the existence of an unramified degree d cover of C in which all points in $C(k)$ that are mapped to G by the Abel-Jacobi map, say $\{p_1, \ldots, p_m\}$, split completely. This cover therefore has genus $d(g-1)+1$ and at least dm rational points (see *e.g.,* [1] or [9]).

The algorithm. Let k be a finite field and let $g > 1$ be an integer. Let $\mathcal{F}_g$ be a list of global function fields over k of genus g. (If $g = 2$ or possible 3, and when k is reasonable small, one may let $\mathcal{F}_g$ be all such fields; for bigger genus one may always take some of the hyperelliptic ones of genus g.)

- For each function field $F \in \mathcal{F}_g$, do the following:
- Compute the class group J of F, together with a map ϕ from the places of F to J given by $Q \mapsto [Q - \deg(Q)P]$ for a fixed rational place P.
- Compute the image I of the rational places of F under ϕ.

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- List all subgroups of J. For each of these, compute its index d and the cardinality m of its intersection with I. This gives the existence of an unramified extension of F of degree d, of genus $d(g-1)+1$ and with at least dm rational places.

The implementation

Using MAGMA we implemented the algorithm over $k = \mathbf{F}_q$ where $q = 3, 5, 7, 9, 11$. We did this for all genus 2 curves, except when $q = 9, 11$ when we only went through the fields that can be constructed as $y^2 = p(x)$ where $p \in \mathbf{F}_{11}[x]$ is of degree 5.

Curves over $\mathbf{F}_3$. For $q = 3$ much work has already been done, and we didn't find any improvements in this case.

Curves over $\mathbf{F}_9$. For $q = 9$ there had also been much previous activity, *e.g.,* [1], [2], [5], [10] (see [4] for a complete set of references). In particular, the present method had been used for many different base curves. By doing a systematic search we found 3 (small) improvements of the old table entries. Let α be a primitive element of $\mathbf{F}_9$ satisfying $\alpha^2 - \alpha - 1 = 0$:

- The field defined by $y^2 = x^5 + \alpha^6 x^3 + \alpha^6 x^2 + \alpha^3 x$ over $\mathbf{F}_9$ has class group isomorphic to $\mathbf{Z}/2 \oplus \mathbf{Z}/2 \oplus \mathbf{Z}/34$ and under the isomorphism given by MAGMA there are rational places mapping to $0, (0,0,17), (1,1,17)$ and $(0,1,0)$. Since these points all lie in the subgroup generated by $(0,0,17), (0,1,0)$ and $(1,0,0)$, which has index 17, we get a cover of genus 18 with at least 68 rational places, improving the old entry $[67, 84]$ slightly.
- The field defined by $y^2 = x^5 + \alpha^6 x^4 + \alpha^7 x^3 + 2x^2 + \alpha^5 x + \alpha^2$ has class group isomorphic to $\mathbf{Z}/155$. Under the isomorphism given by MAGMA there are rational places mapping to $0, 31$ and 62. Since these points all lie in the subgroup generated by 31, which has index 31, we get a cover of genus 32 with at least 93 rational places, improving the old entry $[92, 130]$ slightly.
- The field defined by $y^2 = x^5 + \alpha^7 x^3 + 2x^2 + \alpha^2 x$ over $\mathbf{F}_9$ has class group isomorphic to $\mathbf{Z}/2 \oplus \mathbf{Z}/74$. There are rational places mapping to $0, (1, 37)$ and $(0, 37)$. Since these points all lie in the subgroup generated by $(1, 0)$ and $(0, 37)$, which has index 37, we get a cover of genus 38 with at least 111 rational places. The old entry was $[105, 149]$

Curves over $\mathbf{F}_5$. For $q = 5$ there had been some previous work, *e.g.,* [6], [7] and [8], in particular using the present method. However, by doing the systematic search we found 3 improvements of the old table entries:

- The field defined by $y^2 = x^6 - x^5 - x^4 - x^3 - x^2 - x - 1$ over $\mathbf{F}_5$ has class group isomorphic to $\mathbf{Z}/60$ and under the isomorphism given by MAGMA there are rational places mapping to 0,24,48 and 54. Since these points all lie in the subgroup generated by 6, which has index 6, we get a cover of genus 7 with at least 24 rational places. (Old entry $[22, 26]$.)
- The field defined by $y^2 = x^6 - x^5 + x^4 + x^3 + 2x^2 - 2x - 1$ has class group isomorphic to $\mathbf{Z}/8 \oplus \mathbf{Z}/8$ and under the isomorphism given by MAGMA there are rational places mapping to $0, (6,6), (7,3)$ and $(3,7)$. Since these points all lie in the subgroup generated by $(7,3)$, which has index 8, we get a cover of genus 9 with at least 32 rational places. Because 32 already was known to be an upper bound, this cover has exactly 32 rational places. (The best curve previously known had 26 points.)
- The field defined by $y^2 = x^6 - x^4 - x^3 + x^2 + x$ over $\mathbf{F}_5$ has class group isomorphic to $\mathbf{Z}/55$ and under the isomorphism given by MAGMA there are rational places mapping to 0,22 and 33. Since these points all lie in the

subgroup generated by 11, which has index 11, we get a cover of genus 12 with at least 33 rational places. (Old entry $[30, 38]$.)

Curves over $\mathbf{F}_7$ and $\mathbf{F}_{11}$. For $q = 7$ and 11 there where no lower bounds in the tables for genus greater than 4, so all we had to do to make it to the tables was to produce curves with more that $b(g)/\sqrt{2}$ points, where $b(g)$ is the best known upper bound. The results are given in the table below; we give integers g and N such that there exists a genus g curve with at least N points, and then the interval $[b(q, g)/\sqrt{2}, b(q, g)]$, where $b(q, g)$ is a known upper bound for $N_q(g)$.

New entries for the tables over $\mathbf{F}_7$ and $\mathbf{F}_{11}$.

	q=7		q=11	
g	N	$[\frac{b(g)}{\sqrt{2}}, b(g)]$	N	$[\frac{b(g)}{\sqrt{2}}, b(g)]$
4			33	[25,34]
5	24	[20,28]	36	[27,38]
6	25	[23,32]	40	[32,45]
7	30	[26,36]	42	[36,50]
8	35	[27,38]	42	[39,55]
9	32	[29,41]	48	[42,59]
10	36	[32,45]	54	[45,63]
11	40	[34,48]	60	[48,67]
12	44	[37,51]	66	[51,72]
13	48	[39,54]	60	[55,77]
14				
15			70	[61,85]
16	45	[45,63]		
17	64	[47,66]	80	[66,93]
18	51	[49,68]	85	[70,98]
19	54	[51,71]	90	[73,102]
20			76	[75,106]
21	60	[55,77]	80	[78,110]
22	63	[56,79]		
23	66	[58,82]	88	[85,119]
24			92	[87,123]
25	72	[62,87]	96	[90,127]
26				
27	78	[66,93]		
28			108	[98,138]
33			128	[111,156]

The details of the construction are given in the appendix as a MAGMA output. There, for each entry given in this table, a function field of genus 2 together with all the rest of the information required to prove the existence of the the extension is given.

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Appendix

This appendix contains the output from a MAGMA-session, displaying all data necessary to verify the entries in the tables given in this rapport. It is to be read as follows: For every post, first comes an explicit genus 2 function field; on the next lines is its class group; then comes a list with points in the class group whose preimage in the field contains a rational place; then a subgroup of the class group that contains all these points; then the index of this subgroup. Finally, on the last line is a pair of integers: the first is the genus of the unramified extension given by this data, and the second is a lower bound on its number of rational places.

Curves over $\mathbf{F}_7$.

```
Algebraic function field defined over GF(7) by
x^5 + 6*x^4 + 6*x^3 + 5*x^2 + 6*x + 6*y^2 + 6
Abelian Group isomorphic to Z/108
Defined on 1 generator
Relations:
    108*J.1 = 0
[
    0,
    80*J.1,
    72*J.1,
    8*J.1,
    84*J.1,
    104*J.1
]
Abelian Group isomorphic to Z/27
Defined on 1 generator in supergroup J:
    G.1 = 4*J.1
Relations:
    27*G.1 = 0
4
[ 5, 24 ]
----------------------------
```

```
Algebraic function field defined over GF(7) by
x^5 + 6*x^4 + 5*x^3 + 6*x^2 + x + 6*y^2 + 4
Abelian Group isomorphic to Z/130
Defined on 1 generator
Relations:
    130*J.1 = 0
[
    0,
    105*J.1,
    15*J.1,
    20*J.1,
    50*J.1
]
Abelian Group isomorphic to Z/26
Defined on 1 generator in supergroup J:
    G.1 = 5*J.1
Relations:
    26*G.1 = 0
5
[ 6, 25 ]
----------------------------
Algebraic function field defined over GF(7) by
x^5 + 6*x^4 + 6*x^3 + 6*x^2 + 5*x + 6*y^2
Abelian Group isomorphic to Z/2 + Z/2 + Z/18
Defined on 3 generators
Relations:
    2*J.1 = 0
    2*J.2 = 0
    18*J.3 = 0
[
    9*J.3,
    0,
    J.2 + 9*J.3,
    J.2 + 12*J.3,
    J.2 + 6*J.3
]
Abelian Group isomorphic to Z/2 + Z/6
Defined on 2 generators in supergroup J:
    G.1 = J.2
    G.2 = 3*J.3
Relations:
    2*G.1 = 0
    6*G.2 = 0
6
[ 7, 30 ]
----------------------------
Algebraic function field defined over GF(7) by
x^5 + 6*x^4 + 4*x^3 + 5*x^2 + 5*x + 6*y^2 + 1
Abelian Group isomorphic to Z/105
Defined on 1 generator
Relations:
    105*J.1 = 0
```

```
[
    77*J.1,
    0,
    49*J.1,
    14*J.1,
    35*J.1
]
Abelian Group isomorphic to Z/15
Defined on 1 generator in supergroup J:
    G.1 = 7*J.1
Relations:
    15*G.1 = 0
7
[ 8, 35 ]
----------------------------
Algebraic function field defined over GF(7) by
x^5 + 6*x^4 + 6*x^3 + 3*x^2 + 5*x + 6*y^2 + 5
Abelian Group isomorphic to Z/2 + Z/4 + Z/8
Defined on 3 generators
Relations:
    2*J.1 = 0
    4*J.2 = 0
    8*J.3 = 0
[
    J.1,
    0,
    2*J.2,
    J.1 + 2*J.2 + 4*J.3
]
Abelian Group isomorphic to Z/2 + Z/2 + Z/2
Defined on 3 generators in supergroup J:
    G.1 = J.1
    G.2 = 2*J.2
    G.3 = 4*J.3
Relations:
    2*G.1 = 0
    2*G.2 = 0
    2*G.3 = 0
8
[ 9, 32 ]
----------------------------
Algebraic function field defined over GF(7) by
x^5 + 6*x^4 + 6*x^3 + 6*x^2 + 5*x + 6*y^2
Abelian Group isomorphic to Z/2 + Z/2 + Z/18
Defined on 3 generators
Relations:
    2*J.1 = 0
    2*J.2 = 0
    18*J.3 = 0
[
    9*J.3,
    0,
```

```
    J.2 + 9*J.3,
    J.1 + J.2 + 9*J.3
]
Abelian Group isomorphic to Z/2 + Z/2 + Z/2
Defined on 3 generators in supergroup J:
    G.1 = J.1
    G.2 = J.2
    G.3 = 9*J.3
Relations:
    2*G.1 = 0
    2*G.2 = 0
    2*G.3 = 0
9
[ 10, 36 ]
----------------------------
Algebraic function field defined over GF(7) by
x^5 + 6*x^4 + 6*x^3 + 2*x + 6*y^2 + 3
Abelian Group isomorphic to Z/10 + Z/10
Defined on 2 generators
Relations:
    10*J.1 = 0
    10*J.2 = 0
[
    4*J.1,
    0,
    8*J.1,
    9*J.1
]
Abelian Group isomorphic to Z/10
Defined on 1 generator in supergroup J:
    G.1 = 9*J.1
Relations:
    10*G.1 = 0
10
[ 11, 40 ]
----------------------------
Algebraic function field defined over GF(7) by
x^6 + 2*x^5 + 6*x^4 + 4*x^3 + 6*x^2 + x + 6*y^2 + 2
Abelian Group isomorphic to Z/11 + Z/11
Defined on 2 generators
Relations:
    11*J.1 = 0
    11*J.2 = 0
[
    3*J.1 + 4*J.2,
    J.1 + 5*J.2,
    0,
    4*J.1 + 9*J.2
]
Abelian Group isomorphic to Z/11
Defined on 1 generator in supergroup J:
    G.1 = 10*J.1 + 6*J.2
```

```
Relations:
    11*G.1 = 0
11
[ 12, 44 ]
----------------------------
Algebraic function field defined over GF(7) by
x^5 + 6*x^4 + 5*x^3 + 5*x^2 + 4*x + 6*y^2
Abelian Group isomorphic to Z/2 + Z/2 + Z/24
Defined on 3 generators
Relations:
    2*J.1 = 0
    2*J.2 = 0
    24*J.3 = 0
[
    J.1 + 12*J.3,
    0,
    J.2 + 12*J.3,
    J.2
]
Abelian Group isomorphic to Z/2 + Z/2 + Z/2
Defined on 3 generators in supergroup J:
    G.1 = J.1
    G.2 = J.2
    G.3 = 12*J.3
Relations:
    2*G.1 = 0
    2*G.2 = 0
    2*G.3 = 0
12
[ 13, 48 ]
----------------------------
Algebraic function field defined over GF(7) by
x^5 + 6*x^4 + 5*x^3 + 3*x^2 + 3*x + 6*y^2 + 4
Abelian Group isomorphic to Z/105
Defined on 1 generator
Relations:
    105*J.1 = 0
[
    0,
    60*J.1,
    75*J.1
]
Abelian Group isomorphic to Z/7
Defined on 1 generator in supergroup J:
    G.1 = 15*J.1
Relations:
    7*G.1 = 0
15
[ 16, 45 ]

----------------------------
Algebraic function field defined over GF(7) by
```

```
x^6 + 4*x^5 + 4*x^4 + 4*x^3 + 4*x^2 + 4*x + 6*y^2 + 4
Abelian Group isomorphic to Z/12 + Z/12
Defined on 2 generators
Relations:
    12*J.1 = 0
    12*J.2 = 0
[
    0,
    8*J.1,
    4*J.2,
    8*J.1 + 8*J.2
]
Abelian Group isomorphic to Z/3 + Z/3
Defined on 2 generators in supergroup J:
    G.1 = 4*J.1
    G.2 = 4*J.2
Relations:
    3*G.1 = 0
    3*G.2 = 0
16
[ 17, 64 ]
----------------------------
Algebraic function field defined over GF(7) by
x^5 + 6*x^4 + 3*x^3 + x^2 + 6*x + 6*y^2
Abelian Group isomorphic to Z/2 + Z/34
Defined on 2 generators
Relations:
    2*J.1 = 0
    34*J.2 = 0
[
    J.1,
    0,
    17*J.2
]
Abelian Group isomorphic to Z/2 + Z/2
Defined on 2 generators in supergroup J:
    G.1 = J.1
    G.2 = 17*J.2
Relations:
    2*G.1 = 0
    2*G.2 = 0
17
[ 18, 51 ]
----------------------------
Algebraic function field defined over GF(7) by
x^5 + 6*x^4 + 6*x^3 + 6*x^2 + 5*x + 6*y^2
Abelian Group isomorphic to Z/2 + Z/2 + Z/18
Defined on 3 generators
Relations:
    2*J.1 = 0
    2*J.2 = 0
    18*J.3 = 0
```

```
[
    9*J.3,
    0,
    J.1 + J.2 + 9*J.3
]
Abelian Group isomorphic to Z/2 + Z/2
Defined on 2 generators in supergroup J:
    G.1 = J.1 + J.2
    G.2 = 9*J.3
Relations:
    2*G.1 = 0
    2*G.2 = 0
18
[ 19, 54 ]
----------------------------
Algebraic function field defined over GF(7) by
x^5 + 6*x^4 + 6*x^3 + 2*x + 6*y^2 + 3
Abelian Group isomorphic to Z/10 + Z/10
Defined on 2 generators
Relations:
    10*J.1 = 0
    10*J.2 = 0
[
    4*J.1,
    0,
    8*J.1
]
Abelian Group isomorphic to Z/5
Defined on 1 generator in supergroup J:
    G.1 = 8*J.1
Relations:
    5*G.1 = 0
20
[ 21, 60 ]
----------------------------
Algebraic function field defined over GF(7) by
x^5 + 6*x^4 + 6*x^3 + 4*x^2 + 6*x + 6*y^2
Abelian Group isomorphic to Z/2 + Z/42
Defined on 2 generators
Relations:
    2*J.1 = 0
    42*J.2 = 0
[
    J.1,
    0,
    J.1 + 21*J.2
]
Abelian Group isomorphic to Z/2 + Z/2
Defined on 2 generators in supergroup J:
    G.1 = J.1
    G.2 = 21*J.2
Relations:
```

```
    2*G.1 = 0
    2*G.2 = 0
21
[ 22, 63 ]
----------------------------
Algebraic function field defined over GF(7) by
x^5 + 6*x^4 + 6*x^3 + 2*x^2 + x + 6*y^2
Abelian Group isomorphic to Z/2 + Z/44
Defined on 2 generators
Relations:
    2*J.1 = 0
    44*J.2 = 0
[
    22*J.2,
    0,
    J.1
]
Abelian Group isomorphic to Z/2 + Z/2
Defined on 2 generators in supergroup J:
    G.1 = J.1
    G.2 = 22*J.2
Relations:
    2*G.1 = 0
    2*G.2 = 0
22
[ 23, 66 ]
----------------------------
Algebraic function field defined over GF(7) by
x^5 + 6*x^4 + 5*x^3 + 5*x^2 + 4*x + 6*y^2
Abelian Group isomorphic to Z/2 + Z/2 + Z/24
Defined on 3 generators
Relations:
    2*J.1 = 0
    2*J.2 = 0
    24*J.3 = 0
[
    0,
    J.2 + 12*J.3,
    J.2
]
Abelian Group isomorphic to Z/2 + Z/2
Defined on 2 generators in supergroup J:
    G.1 = J.2
    G.2 = 12*J.3
Relations:
    2*G.1 = 0
    2*G.2 = 0
24
[ 25, 72 ]
----------------------------
Algebraic function field defined over GF(7) by
x^5 + 6*x^4 + 6*x^3 + 3*x^2 + 5*x + 6*y^2
```

```
Abelian Group isomorphic to Z/2 + Z/52
Defined on 2 generators
Relations:
    2*J.1 = 0
    52*J.2 = 0
[
    J.1,
    0,
    J.1 + 26*J.2
]
Abelian Group isomorphic to Z/2 + Z/2
Defined on 2 generators in supergroup J:
    G.1 = J.1
    G.2 = 26*J.2
Relations:
    2*G.1 = 0
    2*G.2 = 0
26
[ 27, 78 ]
----------------------------
```

Curves over $\mathbf{F}_{11}$.

```
Algebraic function field defined over GF(11) by
x^5 + 10*x^4 + 9*x^3 + 4*x^2 + 3*x + 10*y^2 + 4
Abelian Group isomorphic to Z/270
Defined on 1 generator
Relations:
    270*J.1 = 0
[
    3*J.1,
    0,
    6*J.1,
    72*J.1,
    204*J.1,
    252*J.1,
    24*J.1,
    81*J.1,
    195*J.1,
    165*J.1,
    111*J.1
]
Abelian Group isomorphic to Z/90
Defined on 1 generator in supergroup J:
    G.1 = 3*J.1
Relations:
    90*G.1 = 0
3
[ 4, 33 ]
-----------
Algebraic function field defined over GF(11) by
x^5 + 10*x^4 + 9*x^3 + x^2 + 2*x + 10*y^2 + 1
Abelian Group isomorphic to Z/220
Defined on 1 generator
```

```
Relations:
    220*J.1 = 0
[
    0,
    200*J.1,
    204*J.1,
    216*J.1,
    164*J.1,
    36*J.1,
    100*J.1,
    172*J.1,
    28*J.1
]
Abelian Group isomorphic to Z/55
Defined on 1 generator in supergroup J:
    G.1 = 4*J.1
Relations:
    55*G.1 = 0
4
[ 5, 36 ]
-----------
Algebraic function field defined over GF(11) by
x^5 + 10*x^4 + 10*x^3 + 2*x^2 + 8*x + 10*y^2
Abelian Group isomorphic to Z/2 + Z/110
Defined on 2 generators
Relations:
    2*J.1 = 0
    110*J.2 = 0
[
    55*J.2,
    0,
    20*J.2,
    90*J.2,
    J.1 + 70*J.2,
    J.1 + 40*J.2,
    J.1 + 85*J.2,
    J.1 + 25*J.2
]
Abelian Group isomorphic to Z/2 + Z/22
Defined on 2 generators in supergroup J:
    G.1 = J.1
    G.2 = 5*J.2
Relations:
    2*G.1 = 0
    22*G.2 = 0
5
[ 6, 40 ]
-----------
Algebraic function field defined over GF(11) by
x^5 + 10*x^4 + 10*x^3 + 5*x^2 + 5*x + 10*y^2 + 6
Abelian Group isomorphic to Z/222
Defined on 1 generator
```

```
Relations:
    222*J.1 = 0
[
    0,
    144*J.1,
    78*J.1,
    54*J.1,
    168*J.1,
    126*J.1,
    96*J.1
]
Abelian Group isomorphic to Z/37
Defined on 1 generator in supergroup J:
    G.1 = 6*J.1
Relations:
    37*G.1 = 0
6
[ 7, 42 ]
-----------
Algebraic function field defined over GF(11) by
x^5 + 10*x^4 + 10*x^3 + 3*x^2 + 4*x + 10*y^2 + 3
Abelian Group isomorphic to Z/14 + Z/14
Defined on 2 generators
Relations:
    14*J.1 = 0
    14*J.2 = 0
[
    9*J.1 + 3*J.2,
    0,
    4*J.1 + 6*J.2,
    12*J.1 + 11*J.2,
    6*J.1 + 9*J.2,
    9*J.1 + 10*J.2
]
Abelian Group isomorphic to Z/2 + Z/14
Defined on 2 generators in supergroup J:
    G.1 = 7*J.2
    G.2 = 13*J.1 + 2*J.2
Relations:
    2*G.1 = 0
    14*G.2 = 0
7
[ 8, 42 ]
-----------
Algebraic function field defined over GF(11) by
x^5 + 10*x^4 + 10*x^3 + 9*x^2 + 6*x + 10*y^2 + 3
Abelian Group isomorphic to Z/160
Defined on 1 generator
Relations:
    160*J.1 = 0
[
    152*J.1,
```

```
    0,
    144*J.1,
    72*J.1,
    40*J.1,
    104*J.1
]
Abelian Group isomorphic to Z/20
Defined on 1 generator in supergroup J:
    G.1 = 8*J.1
Relations:
    20*G.1 = 0
8
[ 9, 48 ]
-----------
Algebraic function field defined over GF(11) by
x^5 + 10*x^4 + 10*x^3 + 4*x^2 + 9*x + 10*y^2
Abelian Group isomorphic to Z/2 + Z/2 + Z/54
Defined on 3 generators
Relations:
    2*J.1 = 0
    2*J.2 = 0
    54*J.3 = 0
[
    J.1,
    0,
    J.2 + 18*J.3,
    J.2 + 36*J.3,
    J.2 + 27*J.3,
    27*J.3
]
Abelian Group isomorphic to Z/2 + Z/2 + Z/6
Defined on 3 generators in supergroup J:
    G.1 = J.1
    G.2 = J.2
    G.3 = 9*J.3
Relations:
    2*G.1 = 0
    2*G.2 = 0
    6*G.3 = 0
9
[ 10, 54 ]
-----------
Algebraic function field defined over GF(11) by
x^5 + 10*x^4 + 9*x^3 + 5*x^2 + 7*x + 10*y^2 + 4
Abelian Group isomorphic to Z/10 + Z/20
Defined on 2 generators
Relations:
    10*J.1 = 0
    20*J.2 = 0
[
    7*J.1 + 5*J.2,
    0,
```

```
    4*J.1 + 10*J.2,
    5*J.1 + 5*J.2,
    9*J.1 + 5*J.2,
    7*J.1 + 15*J.2
]
Abelian Group isomorphic to Z/20
Defined on 1 generator in supergroup J:
    G.1 = 9*J.1 + 15*J.2
Relations:
    20*G.1 = 0
10
[ 11, 60 ]
-----------
Algebraic function field defined over GF(11) by
x^5 + 10*x^4 + 6*x^3 + 4*x^2 + 4*x + 10*y^2
Abelian Group isomorphic to Z/2 + Z/2 + Z/2 + Z/22
Defined on 4 generators
Relations:
    2*J.1 = 0
    2*J.2 = 0
    2*J.3 = 0
    22*J.4 = 0
[
    J.1 + J.2,
    0,
    J.1 + J.3,
    11*J.4,
    J.3 + 11*J.4,
    J.2
]
Abelian Group isomorphic to Z/2 + Z/2 + Z/2 + Z/2
Defined on 4 generators in supergroup J:
    G.1 = J.1
    G.2 = J.2
    G.3 = J.3
    G.4 = 11*J.4
Relations:
    2*G.1 = 0
    2*G.2 = 0
    2*G.3 = 0
    2*G.4 = 0
11
[ 12, 66 ]
-----------
Algebraic function field defined over GF(11) by
x^5 + 10*x^4 + 10*x^3 + 10*x^2 + 10*x + 10*y^2 + 9
Abelian Group isomorphic to Z/2 + Z/2 + Z/6 + Z/6
Defined on 4 generators
Relations:
    2*J.1 = 0
    2*J.2 = 0
    6*J.3 = 0
```

```
    6*J.4 = 0
[
    J.1 + 5*J.4,
    0,
    4*J.4,
    J.1 + 2*J.4,
    5*J.4
]
Abelian Group isomorphic to Z/2 + Z/6
Defined on 2 generators in supergroup J:
    G.1 = J.1
    G.2 = J.4
Relations:
    2*G.1 = 0
    6*G.2 = 0
12
[ 13, 60 ]
-----------
Algebraic function field defined over GF(11) by
x^5 + 10*x^4 + 8*x^3 + 10*x^2 + 6*x + 10*y^2 + 7
Abelian Group isomorphic to Z/14 + Z/14
Defined on 2 generators
Relations:
    14*J.1 = 0
    14*J.2 = 0
[
    0,
    4*J.1 + 4*J.2,
    2*J.1 + 9*J.2,
    10*J.1 + 3*J.2,
    8*J.1 + J.2
]
Abelian Group isomorphic to Z/14
Defined on 1 generator in supergroup J:
    G.1 = 12*J.1 + 5*J.2
Relations:
    14*G.1 = 0
14
[ 15, 70 ]
-----------
Algebraic function field defined over GF(11) by
x^5 + 10*x^4 + 10*x^3 + 6*x^2 + 5*x + 10*y^2 + 5
Abelian Group isomorphic to Z/2 + Z/8 + Z/16
Defined on 3 generators
Relations:
    2*J.1 = 0
    8*J.2 = 0
    16*J.3 = 0
[
    7*J.2,
    0,
    6*J.2,
```

```
    J.1 + 7*J.2,
    J.1 + 3*J.2
]
Abelian Group isomorphic to Z/2 + Z/8
Defined on 2 generators in supergroup J:
    G.1 = J.1
    G.2 = J.2
Relations:
    2*G.1 = 0
    8*G.2 = 0
16
[ 17, 80 ]
-----------
Algebraic function field defined over GF(11) by
x^5 + 10*x^4 + 6*x^3 + 6*x^2 + 6*x + 10*y^2
Abelian Group isomorphic to Z/2 + Z/102
Defined on 2 generators
Relations:
    2*J.1 = 0
    102*J.2 = 0
[
    51*J.2,
    0,
    J.1 + 17*J.2,
    J.1 + 85*J.2,
    J.1
]
Abelian Group isomorphic to Z/2 + Z/6
Defined on 2 generators in supergroup J:
    G.1 = J.1
    G.2 = 17*J.2
Relations:
    2*G.1 = 0
    6*G.2 = 0
17
[ 18, 85 ]
-----------
Algebraic function field defined over GF(11) by
x^5 + 10*x^4 + 10*x^3 + 4*x^2 + 9*x + 10*y^2
Abelian Group isomorphic to Z/2 + Z/2 + Z/54
Defined on 3 generators
Relations:
    2*J.1 = 0
    2*J.2 = 0
    54*J.3 = 0
[
    0,
    J.2 + 18*J.3,
    J.2 + 36*J.3,
    J.2 + 27*J.3,
    27*J.3
]
```

```
Abelian Group isomorphic to Z/2 + Z/6
Defined on 2 generators in supergroup J:
    G.1 = J.2
    G.2 = 9*J.3
Relations:
    2*G.1 = 0
    6*G.2 = 0
18
[ 19, 90 ]
-----------
Algebraic function field defined over GF(11) by
x^5 + 10*x^4 + 10*x^3 + 5*x^2 + 8*x + 10*y^2 + 2
Abelian Group isomorphic to Z/152
Defined on 1 generator
Relations:
    152*J.1 = 0
[
    95*J.1,
    0,
    38*J.1,
    19*J.1
]
Abelian Group isomorphic to Z/8
Defined on 1 generator in supergroup J:
    G.1 = 19*J.1
Relations:
    8*G.1 = 0
19
[ 20, 76 ]
-----------
Algebraic function field defined over GF(11) by
x^5 + 10*x^4 + 10*x^3 + 8*x^2 + 10*x + 10*y^2 + 6
Abelian Group isomorphic to Z/2 + Z/2 + Z/40
Defined on 3 generators
Relations:
    2*J.1 = 0
    2*J.2 = 0
    40*J.3 = 0
[
    J.1,
    0,
    J.2 + 20*J.3,
    J.1 + 20*J.3
]
Abelian Group isomorphic to Z/2 + Z/2 + Z/2
Defined on 3 generators in supergroup J:
    G.1 = J.1
    G.2 = J.2
    G.3 = 20*J.3
Relations:
    2*G.1 = 0
    2*G.2 = 0
```

```
    2*G.3 = 0
20
[ 21, 80 ]
-----------
Algebraic function field defined over GF(11) by
x^5 + 10*x^4 + 10*x^3 + 7*x^2 + 10*x + 10*y^2
Abelian Group isomorphic to Z/2 + Z/2 + Z/44
Defined on 3 generators
Relations:
    2*J.1 = 0
    2*J.2 = 0
    44*J.3 = 0
[
    J.1,
    0,
    22*J.3,
    J.1 + J.2
]
Abelian Group isomorphic to Z/2 + Z/2 + Z/2
Defined on 3 generators in supergroup J:
    G.1 = J.1
    G.2 = J.2
    G.3 = 22*J.3
Relations:
    2*G.1 = 0
    2*G.2 = 0
    2*G.3 = 0
22
[ 23, 88 ]
-----------
Algebraic function field defined over GF(11) by
x^5 + 10*x^4 + 10*x^3 + 8*x^2 + 8*x + 10*y^2 + 5
Abelian Group isomorphic to Z/184
Defined on 1 generator
Relations:
    184*J.1 = 0
[
    115*J.1,
    0,
    46*J.1,
    23*J.1
]
Abelian Group isomorphic to Z/8
Defined on 1 generator in supergroup J:
    G.1 = 23*J.1
Relations:
    8*G.1 = 0
23
[ 24, 92 ]
-----------
Algebraic function field defined over GF(11) by
x^5 + 10*x^4 + 10*x^3 + 6*x^2 + 4*x + 10*y^2
```

```
Abelian Group isomorphic to Z/192
Defined on 1 generator
Relations:
    192*J.1 = 0
[
    96*J.1,
    0,
    24*J.1,
    168*J.1
]
Abelian Group isomorphic to Z/8
Defined on 1 generator in supergroup J:
    G.1 = 24*J.1
Relations:
    8*G.1 = 0
24
[ 25, 96 ]
-----------
Algebraic function field defined over GF(11) by
x^5 + 10*x^4 + 10*x^3 + 4*x^2 + 9*x + 10*y^2
Abelian Group isomorphic to Z/2 + Z/2 + Z/54
Defined on 3 generators
Relations:
    2*J.1 = 0
    2*J.2 = 0
    54*J.3 = 0
[
    J.1,
    0,
    J.2 + 27*J.3,
    27*J.3
]
Abelian Group isomorphic to Z/2 + Z/2 + Z/2
Defined on 3 generators in supergroup J:
    G.1 = J.1
    G.2 = J.2
    G.3 = 27*J.3
Relations:
    2*G.1 = 0
    2*G.2 = 0
    2*G.3 = 0
27
[ 28, 108 ]
-----------
Algebraic function field defined over GF(11) by
x^5 + 10*x^4 + 10*x^3 + 6*x^2 + 5*x + 10*y^2 + 5
Abelian Group isomorphic to Z/2 + Z/8 + Z/16
Defined on 3 generators
Relations:
    2*J.1 = 0
    8*J.2 = 0
    16*J.3 = 0
```

```
[
    0,
    6*J.2,
    J.1 + 7*J.2,
    J.1 + 3*J.2
]
Abelian Group isomorphic to Z/8
Defined on 1 generator in supergroup J:
    G.1 = J.1 + J.2
Relations:
    8*G.1 = 0
32
[ 33, 128 ]
-----------
```

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